The odd writhe is an obstruction to sliceness

4th Russian-Chinese Conference on Knot Theory and Related Topics

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Preliminaries

Introduction

In this talk we shall prove the following theorem.

Theorem

Let K be a virtual knot and $J(K)$ its odd writhe. If $J(K) \neq 0$ then K is not slice.

The proof uses *doubled Khovanov homology*, an extension of Khovanov homology to virtual links. Specifically, a perturbation of doubled Khovanov homology yields the *doubled Rasmussen invariant*. In order to prove the above theorem we shall show that the doubled Rasmussen invariant contains the odd writhe, and that it is a concordance invariant.

To begin, we shall quickly recall the definition of the odd writhe, and give an overview of the construction of doubled Khovanov homology.

Much of the content of this talk is contained in the paper *Doubled Khovanov homology*, available at `front.math.ucdavis.edu/1704.07324`.

Preliminaries

The odd writhe

The odd writhe (Kauffman 2004)

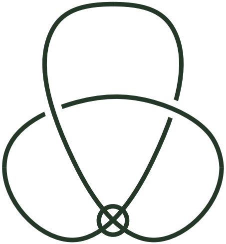
Definition

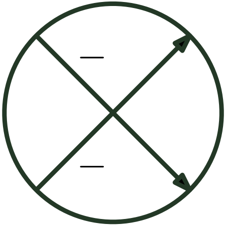
Let D be a diagram of a virtual knot and $G(D)$ its Gauss diagram. A classical crossing of D, associated to the chord labelled c in $G(D)$, is known as *odd* if the number of chord endpoints appearing between the two endpoints of c is odd. Otherwise it is known as *even*. The *odd writhe* of D is defined

$$J(D) = \sum_{\text{odd crossings of } D} \text{sign of the crossing.}$$

Theorem

Let D be a virtual knot diagram of K. The odd writhe is an invariant of K and we define $J(K) := J(D)$.





For example, consider the above diagram of virtual knot 2.1 and its Gauss code. We see that $J(2.1) = -2$.

The odd writhe is easy to compute and provides an obstruction to classicality (and hence triviality) and amphichirality of virtual knots.

Regarding classicality, one easily sees that a classical knot necessarily has trivial odd writhe. The unknot is a classical knot, so if a virtual knot has non-zero odd writhe it is non-classical and hence non-trivial.

Regarding amphichirality, it can be seen that $J(K) = -J(-K)$, where $-K$ denotes the mirror image of K.

Preliminaries

Doubled Khovanov homology

Doubled Khovanov homology (R. 2017)

Khovanov homology is a powerful invariant of classical links. Using $\mathbb{Z}_2$ coefficients one can apply it, without modification, to virtual links. If one wishes to use arbitrary coefficients, however, some work must be done.

Manturov first extended Khovanov homology with arbitrary coefficients to virtual links. His work was reformulated by Dye, Kaestner, and Kauffman, in order to - among other things - produce a virtual Rasmussen invariant. Tubbenhauer has also produced a virtual Khovanov theory in the manner of Bar-Natan (albeit with compatibility issues with the other theories).

Doubled Khovanov homology is an alternative extension of Khovanov homology to virtual links. It differs from the other extensions in that it does not use any new diagrammatic technology; the work is done in algebra.

In a nutshell, doubled Khovanov homology is constructed by "doubling up" the module assigned to a circle. It turns out that this allows one to define a homology theory (with a caveat that we are no longer applying a topological quantum field theory (TQFT)).

We will not give a full definition of doubled Khovanov homology, focussing instead on the perturbation of the theory known as *doubled Lee homology*. Here we simply fix some notation.

Definition

Let L be a virtual link. Denote by $DKh(L)$ the doubled Khovanov homology of L, a bigraded finitely generated Abelian group.

The doubled Rasmussen invariant

Doubled Khovanov homology can be used to produce a concordance invariant of virtual knots, the *doubled Rasmussen invariant*. Before defining this invariant, we shall describe a perturbation of doubled Khovanov homology.

The doubled Rasmussen invariant

Doubled Lee homology

Perturbing doubled Khovanov homology

There is a perturbation of classical Khovanov homology due to Lee; it is formed by adding a term of quantum grading $+4$ to the differential. This technique may be repeated with doubled Khovanov homology, yielding *doubled Lee homology*.

Definition

Let L be a virtual link. Denote by $DKh'(L)$ the doubled Lee homology of L. Denote by $DKh'_k(L)$ the homogenous elements of homological degree k.

Doubled Lee homology shares some properties with its classical counterpart, but it also exhibits some important differences.

As in the classical theory, generators of $DKh'(L)$ are associated to alternately coloured smoothings of L. Henceforth we shall use the shorthand ACS = alternately coloured smoothing.

Theorem

Given a virtual link L

$$rank\left(DKh'(L)\right) = 2\left|\{ACSs\ of\ L\}\right|$$

Pick a diagram D of L, and let $\mathscr{S}$ be an ACS of D. Then

$$\mathscr{S} \longleftrightarrow \mathfrak{s}^{\mathsf{u}}, \mathfrak{s}^{\mathsf{l}}$$

where $\mathfrak{s}^{\mathsf{u}}, \mathfrak{s}^{\mathsf{l}} \in DKh'(L)$ are generators of $DKh'(L)$.

There are two gradings on $DKh'(L)$: the homological grading i, and the quantum grading j (both familiar from classical Khovanov homology). The homological grading of $\mathfrak{s}^{\mathrm{u}}$ and $\mathfrak{s}^{\mathrm{l}}$ is easy to compute from $\mathscr{S}$:

$$i(\mathfrak{s}^{\mathrm{u}}) = i(\mathfrak{s}^{\mathrm{l}}) = \text{ number of 1-resolutions in } \mathscr{S} - n_-$$

where n_- denotes the number of negative classical crossings of D. The quantum grading is harder to compute.

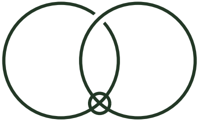
The doubled Rasmussen invariant

Alternately coloured smoothings of virtual links

How many ACSs does a virtual link have?

Combining the fact that $DKh'(L)$ is an invariant of the virtual link L with the above theorem relating generators of $DKh'(L)$ with ACSs, we see that the number of ACSs is a link invariant.

A classical link diagram has one ACS for each of its orientations. This is no longer the case for virtual knots; consider the following virtual link diagram:



Neither of its smoothings are alternately colourable, and its doubled Lee homology is trivial i.e. DKh' () $= 0$.

In light of this, we must ask how many ACSs a virtual link has, in general. We answer this question using Gauss diagrams.

Definition

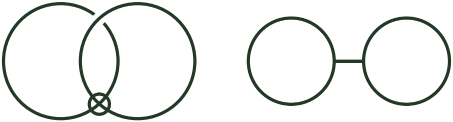
A circle within a Gauss diagram is *degenerate* if it contains an odd number of chord endpoints.

Theorem

Given a diagram D of a virtual link L

$$\begin{aligned}|\{\mathit{ACSs\ of\ } L\}| &= |\{\mathit{ACSs\ of\ } D\}| \\ &= \begin{cases} 2^{|L|}, & \mathit{if\ } G(D) \mathit{\ has\ no\ degenerate\ circles} \\ 0, & \mathit{otherwise.} \end{cases}\end{aligned}$$

This explains why the diagram given above has no ACSs: both of the circles in its Gauss diagram are degenerate.



Corollary

Every virtual knot K has 2 ACSs, so that

$$rank(DKh'(K)) = 4.$$

To prove the corollary one simply needs to observe that as the Gauss diagram of a knot contains only one circle, and each chord has two endpoints, this single circle cannot be degenerate.

The doubled Rasmussen invariant

Defining the invariant

The above corollary shows us that the doubled Lee homology of a virtual knot is always of rank 4. Further, it can be shown that the two ACSs of a virtual knot always share the same underlying smoothing. That is, they can be obtained from one another by simply swapping the colours of the circles.

Given diagram D of a virtual knot K, let $\mathscr{S}$ be an ACS of D, and $\overline{\mathscr{S}}$ denote the ACS obtained by flipping the colouring (i.e. sending red to green and green to red). It is clear that both $\mathscr{S}$ and $\overline{\mathscr{S}}$ are at the same height, so that there is only one non-trivial homological degree in $DKh'(K)$.

Further, it turns out that if the quantum degree of any of the 4 generators of $DKh'(K)$ is known, one can recover the quantum degree of all the others. Thus the information contained in the non-trivial quantum degrees of $DKh'(K)$ is equivalent to a single integer, and we can make the following definition.

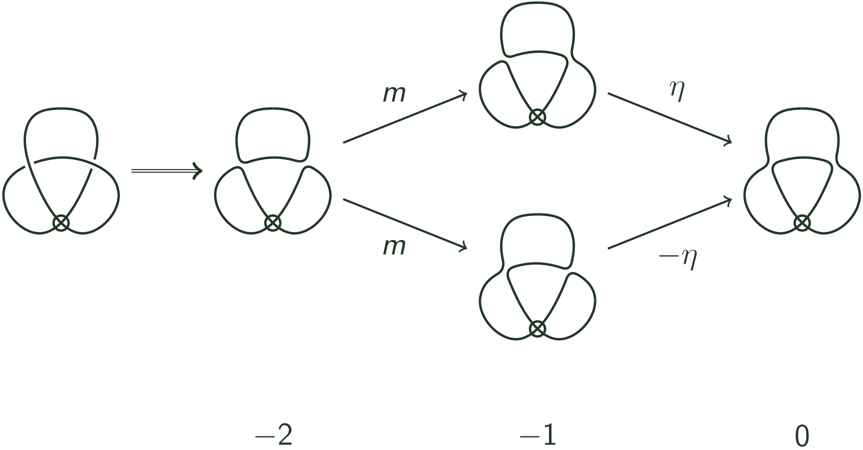
Definition

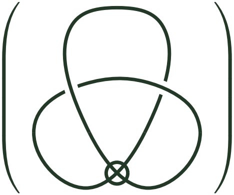
For a virtual knot K let $\mathbb{s}(K) = (s_1(K), s_2(K)) \in \mathbb{Z} \times \mathbb{Z}$ where

$$s_1(K) = \text{ highest non-trivial } j \text{ grading of } DKh'(K) \ - 1$$
$$s_2(K) = i(\mathfrak{s}^{\mathsf{u/l}}) = \text{ height of } \ \mathscr{S} \text{ or } \overline{\mathscr{S}}.$$

We refer to $\mathbb{s}(K)$ as the *doubled Rasmussen invariant* of K.

An example



DKh'  $=$

$j \backslash i$	-2	-1	0
-1			$\cancel{\mathbb{Z}}$
-2			
-3			$\cancel{\mathbb{Z}_2}$
-4	$\mathbb{Z}$		
-5	$\mathbb{Z}$	$\cancel{\mathbb{Z}}$	
-6	$\mathbb{Z}$		
-7	$\mathbb{Z}$		

$$\mathbb{s}(2.1) = (-5, -2)$$

The doubled Rasmussen invariant

$s_2(K) = J(K)$

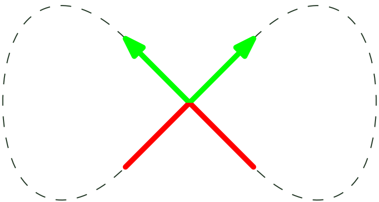
Theorem

Let be D a diagram of a virtual knot K. Then $s_2(K) = J(K)$.

Recall that $s_2(D)$ is equal to the height of the alternately coloured smoothings of D. Further, recall that - in contrast to the classical case - the oriented smoothing of a virtual knot diagram is not alternately colourable, in general. In fact, the alternately colourable smoothing of virtual knot 2.1 is the *unoriented smoothing*.

Therefore, in order to prove this theorem, we shall show that a classical crossing of a knot diagram D is even if and only if it is resolved into its oriented resolution in the ACSs of D.

To see this, consider the following situation at a classical crossing of D:



Next, consider the table of contributions of each crossing to both $s_2(K)$ and $J(K)$:

sign	parity	reso.	$J(K)$	$s_2(K)$
$+$	odd	1	$+1$	$+1$
$+$	even	0	0	0
$-$	odd	0	-1	-1
$-$	even	1	0	0

Interaction with cobordisms

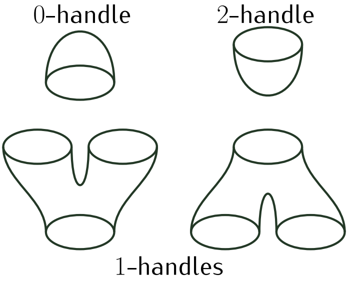
We move on to looking at how the doubled Rasmussen invariant $\mathbb{s}(K)$ interacts with cobordisms between virtual knots. In particular we shall look at its interaction with concordances.

Our aim is to show that a concordance S between two virtual knots K_1 and K_2 induces a map $\phi_S : DKh'(K_1) \rightarrow DKh'(K_2)$, and that this map is necessarily non-zero.

Interaction with cobordisms

Assigning maps to cobordisms

As in the case of classical Khovanov homology, we can assign maps on DKh' to cobordisms. We do this by breaking cobordisms down into *elementary cobordisms*: they correspond to the virtual Reidemeister moves, and 0-, 1- , and 2-handle additions. The elementary cobordisms associated to the virtual Reidemeister moves are all topologically (disjoint unions of) cylinders. The cobordisms associated to handles are as follows:



We will not give a full definition of the maps assigned to elementary cobordisms, moving straight on to the definition of the map assigned to a general cobordism. Let us simply say that every elementary cobordism is assigned a well defined map on doubled Lee homology.

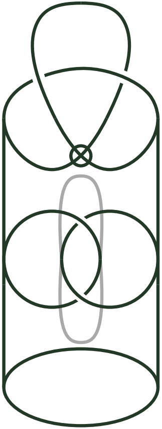
Let S be a cobordism between virtual links L and L'. Further, let

$$S = S_1 \cup S_2 \cup \cdots \cup S_n$$

where S_i is an elementary cobordism (we can always do this, perhaps after an isotopy). Define $\phi_S : DKh'(L) \to DKh'(L')$ to be

$$\phi_S = \phi_{S_n} \circ \phi_{S_{n-1}} \circ \cdots \phi_{S_2} \circ \phi_{S_1}.$$

An example



This cobordism, between virtual knot 2.1 and the unknot, is made up of two 1-handle additions. The first splits one component into two, the next merges two components into one; it describes a surface of genus 1.

This cobordism must be assigned the zero map, however: the maps ϕ_S are necessarily homological degree preserving, and we observed above that $s_2(2.1) = -2$, while $s_2(\text{unknot}) = 0$. Thus the only homological degree preserving map between $DKh'(2.1)$ and $DKh'(\text{unknot})$ is the zero map.

Unlike in the classical case, many connected cobordisms must be assigned the zero map, a consequence of one of the following two phenomena:

1. if there does not exist a homological degree k such that both ${DKh'}_k(L)$ and ${DKh'}_k(L')$ are non-trivial, there can be no map $\phi : DKh'(L) \to DKh'(L')$ which is both non-zero and homological degree preserving
2. if a cobordism S is such that a link L appears within it with $DKh'(L) = 0$, then ϕ_S is the zero map

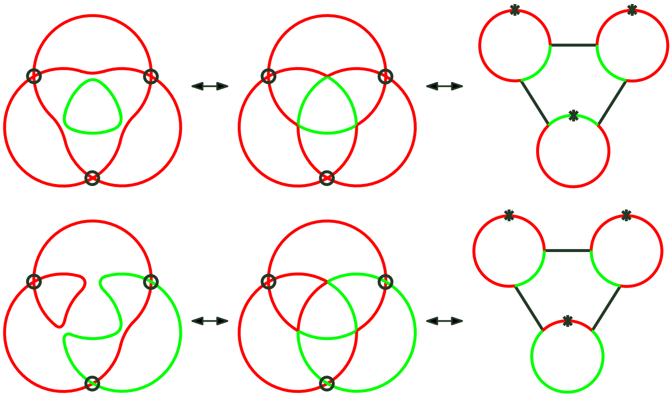
In order to make use of cobordisms between virtual links we must identify classes of cobordisms which are assigned non-zero maps.

As we saw above, whether or not a virtual link possesses ACSs or not depends on the cirlces within its Gauss diagram. It is possible for a 1-handle which splits one component into two to create a degenerate circle, and thus destroy all ACSs.

Theorem

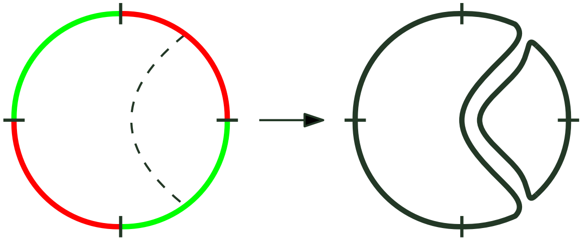
Let S be an elementary cobordism between virtual links L and L'. Further assume S is a 1-handle addition which splits one component into two. If S is attached between arcs of L which are coloured opposite colours in an ACS of L, then L' has no ACSs and $DKh'(L') = 0$.

In order to prove this theorem we must first outline a bijective correspondence between ASCs, colourings of shadows, and colourings of Gauss diagrams:

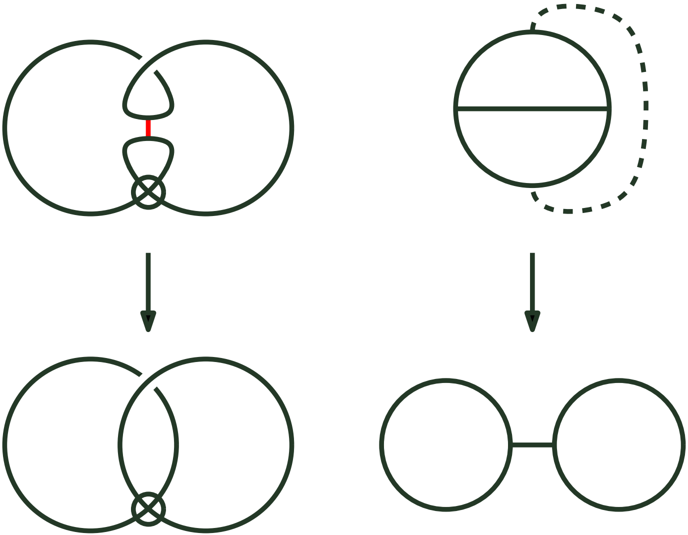


Thus we have a bijective correspondence between ACSs and particular colourings of Gauss diagrams. One sees that if a circle in a Gauss diagram is degenerate, it cannot be coloured in the manner described. Thus if a 1-handle addition creates a degenerate circle, it prohibits the existence of ACSs.

Below we demonstrate how a 1-handle may create a degenerate circle:



An example



To recap, a connected cobordism between virtual links must be assigned the zero map if the homological degrees of the initial and terminal links do not match up, or if a link appears in the cobordism which has no ACSs.

Our final task is to show that a concordance between knots - a cobordism of genus 0 - does not suffer from the above problems, and is assigned a non-zero map.

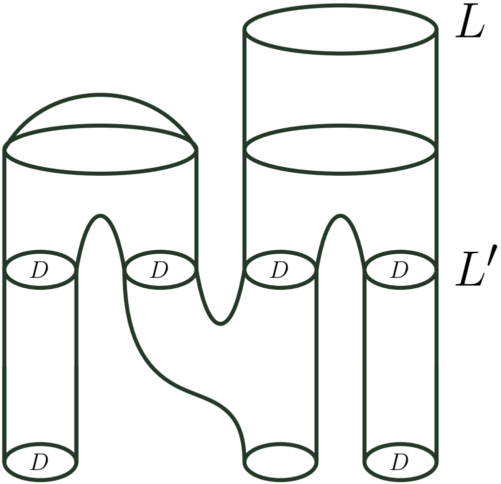
Theorem

Let S be a concordance between virtual knots K and K'. Then $\phi_S : DKh'(K) \to DKh'(K')$ is non-zero.

The proof of this theorem proceeds as follows: first, it is shown that every link appearing in a concordance between knots has ACSs, then an induction argument is used.

To this first point, notice that degenerate circles are always created in pairs, and that one degenerate circle can be canceled against another to produce a non-degenerate circle. However, if two degenerate circles which lie on the same connected component are canceled together, a piece of genus is introduced. This cannot occur in a concordance, of course, so that circles from different connected components must be canceled together. One sees that this leads to a non-compact situation.

Degenerate circles within cobordisms



The proof is concluded using an induction argument on the number of elementary cobordisms making up a concordance. The idea is to start with a concordance which is assumed to be assigned a non-zero map, and add on an elementary cobordism. One then shows that the resulting concordance is also assigned a non-zero map.

Conclusions

Proving the main theorem

Armed with the result from the previous section, we can quickly complete the proof that the odd writhe is an obstruction to sliceness.

Proof of the main theorem.

Let K and K' be concordant virtual knots, and S a concordance between them. Then, as observed above, $\phi_S : DKh'(K) \to DKh'(K')$ is non-zero, and homological degree preserving. As K is a virtual knot, the only non-trivial part of $DKh'(K)$ is $DKh'_{s_2(K)}(K)$. But K' is also a knot, so $DKh'_{s_2(K')}(K')$ is the only non-trivial part of its homology. As ϕ_S is non-zero, we must have $0 \neq \phi_S(x) \in DKh'_{s_2(K)}(K')$, so that $s_2(K) = s_2(K')$ and $J(K) = J(K')$. $\square$

Restricting to the case in which K' is the unknot, so that $J(K') = 0$, we see that if K is slice $J(K) = 0$, as required.

As a quick corollary we also obtain the following.

Theorem

Let K be a virtual knot. If $J(K) \neq 0$ then K is not concordant to a classical knot.

Thus we see that the odd writhe partitions concordance classes of virtual knots.